Solutions 12

Exercise 3.8

(a).

$$\begin{aligned} H_0 : f(y_k|H_0) &= \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2}(y_k - m_0)^2\right) \\ H_1 : f(y_k|H_1) &= \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2}(y_k - m_1)^2\right) \\ Z_k = \ln L(Y_k) &= \frac{m_1 - m_0}{\sigma^2} Y_k + \frac{m_0^2 - m_1^2}{2\sigma^2} \\ \Lambda_n = \sum_k^n Zk &= \frac{m_1 - m_0}{\sigma^2} \sum_k^n Y_k + \frac{n(m_0^2 - m_1^2)}{2\sigma^2} \end{aligned}$$

Let $c = \frac{a\sigma^2}{m_1 - m_0}, \eta = \frac{m_0 + m_1}{2}, d = \frac{b\sigma^2}{m_1 - m_0}$, we can obtain

$$c + \eta n < \sum_k^n Y_k < d + \eta n$$

(b).

$P_F = P_M = 10^{-4}, m_1 = 2, m_0 = 1, \sigma^2 = 1,\ a = \ln A = \ln \frac{P_M}{1 - P_F} = \ln \frac{10^{-4}}{1 - 10^{-4}}, b = \ln B = \ln(10^4 - 1)$. Then we have

$$c = \frac{a\sigma^2}{m_1 - m_0} = \ln \frac{10^{-4}}{1 - 10^{-4}}, \eta = \frac{m_0 + m_1}{2} = \frac{3}{2}, d = \frac{b\sigma^2}{m_1 - m_0} = \ln(10^4 - 1).$$

(c).

$$\begin{aligned} E[N|H_0] &= (bP_F + a(1 - P_F))/m_0 \\ E[N|H_1] &= (aP_M + b(1 - P_M))/m_1 \end{aligned}$$

(d).

$$N = -\frac{\ln(P_E)}{C(f_1, f_0)}$$

where $C(f_1, f_0) = \frac{d^2}{8}$, then we can obtain the results compared with the values in part (c).

Exercise 3.9

(a).

$$H_0 : f(y_k|H_0) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2} y_k^2\right)$$
$$H_1 : f(y_k|H_1) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2} y_k -^2\right)$$
$$Z_k = \ln L(Y_k) = (\frac{1}{2\sigma_0^2} - \frac{1}{2\sigma_1^2}) Y_k^2 + \ln \frac{\sigma_0}{\sigma_1}$$
$$\Lambda_n = \sum_k^n Zk = (\frac{1}{2\sigma_0^2} - \frac{1}{2\sigma_1^2}) \sum_k^n Y_k^2 + n \ln \frac{\sigma_0}{\sigma_1}$$

Let $c = \frac{2a}{\frac{1}{\sigma_0^2} - \frac{1}{\sigma_1^2}}, \eta = \frac{2(\ln \sigma_1 - \ln \sigma_0)}{\frac{1}{\sigma_0^2} - \frac{1}{\sigma_1^2}}, d = \frac{2b}{\frac{1}{\sigma_0^2} - \frac{1}{\sigma_1^2}}$, we can obtain

$$c + \eta n < \sum_k^n Y_k^2 < d + \eta n$$

(b).
$A = \frac{1}{10^4 - 1}, B = 10^4 - 1$.
Hence we have

$$c = -4 \ln(10^4 - 1), \eta = 2 \ln 2, d = 4 \ln(10^4 - 1).$$

(c).

$$E[N|H_0] \approx -\frac{\ln(P_M)}{D(f_0|f_1)}$$
$$E[N|H_1] \approx -\frac{\ln(P_M)}{D(f_1|f_0)}$$

(d).

$$N = -\frac{\ln(P_E)}{C(f_1, f_0)}$$

Then we can obtain the results compared with the values in part (c).

Exercise 3.10

(a).

$$L(Y_k) = \begin{cases} 0 & Y_k < L - 1/2 \\ 1 & L - 1/2 \le Y_k < 1/2 \\ \infty & 1/2 \le Y_k < L + 1/2 \end{cases}$$

We select H_0 whenever

$$\prod_{k=1}^n L(Y_k) \le A,$$

H_1 whenever

$$\prod_{k=1}^{n} L(Y_k) \geq B,$$

and we take another sample if

$$A < \prod_{k=1}^{n} L(Y_k) < B.$$

(b).

No, we can not sure when stopping condition is achieved.

(c).

$$E[N|H_0] = (P_F \ln\left(\frac{1-P_M}{P_F}\right) + (1-P_F)\ln\left(\frac{P_M}{1-P_F}\right))/m_0$$
$$E[N|H_1] = (P_M \ln\left(\frac{P_M}{1-P_F}\right)) + (1-P_M)\ln\left(\frac{1-P_M}{P_F}\right))/m_1$$

Exercise 3.11

(a).

$$\begin{aligned}\pi_0(y_1) = P[H_0|Y_1 = y_1] &= \frac{\pi_0 f_0(y_1)}{\pi_0 f_0(y_1) + (1-\pi_0) f_1(y_1)} \\ &= \begin{cases} 1 & Y_1 < 0 \\ \pi_0 & 0 \leq Y_1 < 1/2 \\ 0 & Y_1 > 1/2 \end{cases}\end{aligned}$$

$$\begin{aligned}T(\pi_0) &= \min\{C_F \pi_0, C_M(1-\pi_0)\} \\ &= \begin{cases} \pi_0 & \pi_0 < 2/3 \\ 2(1-\pi_0) & Y_1 > 2/3 \end{cases}\end{aligned}$$

$$V(\pi_0) = \min\{T(\pi_0), D + E_{Y_1}[V(\pi_0(Y_1))]\}$$